Partial Ordering and Stochastic Resonance for Discrete Memoryless Channels

Dr. Paul Cotae
University of the District of Columbia, Washington DC
pcotae@udc.edu
Outline

- Introduction, Motivation and State of the Art
- (2,3) DMC Mutual Information and Capacity
- (2,3) DMC and particular cases
- Partial Ordering of BSC/SE
- Stochastic Resonance in BSC/SE
- Stochastic Resonance and Capacity Bounds
- Conclusions
Introduction

- Information theory associated with communication channels
- Relations between the general (2,3)DMC, the BAC/SE, the BSC/SE, the BSC, and the BEC
- Partial ordering for the BSC/SE under specific noise element constraints
- Stochastic Resonance effect on the BSC/SE
- Optimal noise power level required to obtain a maximum capacity channel for a given threshold decision
Underwater Sensor Network Architecture
Motivation

- Which DMC, for a given set of resources and noise constraints, is “better” for communication?

- How to use noise to increase the capacity of a communication channel?

- This comparison and increase in channel capacity are necessary for wireless sensor communication links at the physical layer operating in very noisy environment and very limited computational power.

- Stochastic Resonance observed in many non-linear systems.

- Consider Helgert and Pinsker limits capacity bounds
Communication Channel

Continuous Channel

$X_i \sim N(0, \sigma^2)$

Discrete Channel

$X \rightarrow y_1, y_2, \ldots, y_n$

$X = \{x_1, x_2, \ldots, x_m\}$

$P_{11}, P_{2m}, \ldots, P_{nm}$
Entropy, Mutual Information and Capacity

\[ H(X) = - \sum_{x \in X} p(x) \log_2(p(x)) \]

\[ I(X; Y) = H(X) - H(Y|X) \]

\[ C(X, Y) = \max_x I(X, Y) \]

\[ H(X,Y) = H(Y,X) = H(X) + H(Y|X) = H(X|Y) + I(X;Y) + H(Y|X) \]
Partial Ordering
Stochastic Resonance (SR)

SR phenomenon is a non-linear effect wherein a communication system can transmit the information with improved efficiency in the presence of additive noise.
State of the Art

\[ M(a, b) = \begin{bmatrix} a & 1-a \\ b & 1-b \end{bmatrix} = \begin{bmatrix} a & \bar{a} \\ b & \bar{b} \end{bmatrix} \]

\[ h(x) = -x \log_2 x - (1-x) \log_2 (1-x) \]

\[ C(a, b) = \begin{cases} \log_2 \left( \frac{\bar{a}h(b) - \bar{b}h(a)}{a-b} \right) + \log_2 \left( 1 + 2 \frac{h(a)-h(b)}{a-b} \right) & a \neq b \\ \log_2 \left( \frac{\bar{a}h(b) - \bar{b}h(a)}{2(a-b)} + 2 \frac{bh(a)-ah(b)}{a-b} \right) & a \neq b \end{cases} \]
Capacity of the (2,2) DMC
Partial Ordering of (2,2) DMCs
SR in (2,2) DMC
(2,3) DMC

\[ M = \begin{pmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \end{pmatrix} \]

\[ P(X = 1) = x = 1 - P(X = -1) \]
(2,3) DMC Mutual Information and Capacity

\[ C(X, Y) = \max_x I(X, Y) \]

\[ C(X, Y) = \max_x \sum_{i=1}^{2} \sum_{j=1}^{3} p(x_i) p(y_j|x_i) \log_2 \frac{p(y_j|x_i)}{\sum_{k=1}^{2} p(x_k) p(y_j|x_k)} \]
Binary Asymmetric Channel with Symmetric Erasure (BAC/SE) Capacity

using the Kuhn-Tucker condition

\[ p_{21} = p_{22} \quad \text{and} \quad p_{11} > p_{12} \]

\[ C(X,Y) = \sum_{j=1}^{3} p(y_j|x) \log_2 \frac{p(y_j|x)}{x(p(y_j|x_1)-p(y_j|x_2))+p(y_j|x_2)} \]

\[ x = \frac{kp_{32}-p_{12}}{p_{11}-p_{12}-k(p_{31}-p_{32})} \]

\[ k = \left( \frac{p_{11}p_{11}p_{31}p_{31}}{p_{12}p_{12}p_{32}p_{32}} \right) \frac{1}{p_{11}-p_{12}} \]
Binary Symmetric Channel with Symmetric Erasure (BSC/SE) Capacity

\[ p_{11} = p_{32} \]

\[ M = \begin{pmatrix} p_{11} & p_{22} & p_{12} \\ p_{12} & p_{22} & p_{11} \end{pmatrix} \]

\[ C = p_{11} \log_2 \frac{2p_{11}}{p_{11} + p_{31}} + p_{31} \log_2 \frac{2p_{31}}{p_{11} + p_{31}} \]
$p_{21} = p_{22} = 0$

$C = 1 - h(p_{11})$

$p_{12} = p_{31} = 0$

$C = p_{11} = 1 - p_{21}$
\[ p_{11} = a \text{ and } p_{31} = b \]
Partial Ordering of BSC/SE

\[ C(a, b) = a \log_2 \frac{2a}{a + b} + b \log_2 \frac{2b}{a + b} \]

with \(1 \geq a, b \geq 0\) and \(a + b \leq 1\)
\[
\frac{\partial C(a, b)}{\partial a} = \log_2 \left( \frac{2a}{a+b} \right) > 0
\]

\[
\frac{\partial C(a, b)}{\partial b} = \log_2 \left( \frac{2b}{a+b} \right) < 0
\]
Partial Ordering of BSC/SE: Result-I

Theorem 4.2.1: For two BSC/SEs, $M_i = \begin{pmatrix} a_i & 1 - a_i - b_i & b_i \\ b_i & 1 - a_i - b_i & a_i \end{pmatrix}$

and $M_1 = \begin{pmatrix} a_1 & 1 - a_1 - b_1 & b_1 \\ b_1 & 1 - a_1 - b_1 & a_1 \end{pmatrix}$, if $a_i < a_1$ and $b_i > b_1$ then $C(a_i, b_i) < C(a_1, b_1)$

$$A = \begin{pmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$$

$0.6 > 0.5 \text{ and } 0.2 < 0.3 \quad \Rightarrow \quad C(A) > C(B)$
Majorization

\[ a = (a_1, a_2, \ldots, a_n) \quad b = (b_1, b_2, \ldots, b_n) \]

\[ \sum_{i=1}^{k} a_i \downarrow \leq \sum_{i=1}^{k} b_i \downarrow \quad k = 1, 2, \ldots, n - 1 \]

\[ \sum_{i=1}^{n} a_i \downarrow = \sum_{i=1}^{n} b_i \downarrow \]

\[ a \prec b \]
Partial Ordering of BSC/SE (cont.)

\[ a+b \text{ constant, and } a_1 < a_2, \text{ then } (a_1, b_1) \text{ is majorized by } (a_2, b_2) \]

\[ \overrightarrow{u} \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \]

\[ \overrightarrow{v} \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \]

\[ D_{\overrightarrow{u}} C(x, y) = \frac{1}{\sqrt{2}} \log_2 \left( \frac{y}{x} \right) < 0 \]

\[ D_{\overrightarrow{v}} C(x, y) = \frac{1}{\sqrt{2}} \log_2 \left( \frac{4xy}{(x+y)^2} \right) < 0 \]
Schur Convexity: Result-II-a

\[ a \prec b \Rightarrow \varphi(a) \leq \varphi(b) \]

**Theorem 4.2.3:** If \( a+b \) is constant, then the BSC/SE is Schur convex

\[ M_1 = \begin{pmatrix} 0.51 & 0.32 & 0.17 \\ 0.17 & 0.32 & 0.51 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0.4 & 0.32 & 0.28 \\ 0.3 & 0.32 & 0.4 \end{pmatrix} \]

\[ 0.51 + 0.17 = 0.4 + 0.28 \]

\[ (0.51, 0.17) > (0.4, 0.28) \]

\[ C(0.51, 0.17) > C(0.4, 0.28) \]
Theorem 4.2.4: for a - b constant, if $a_1 > a_2$, then $C(a_1, b_1) < C(a_2, b_2)$

\[ M_1 = \begin{pmatrix} 0.51 & 0.32 & 0.17 \\ 0.17 & 0.32 & 0.51 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0.62 & 0.1 & 0.28 \\ 0.28 & 0.1 & 0.62 \end{pmatrix} \]

$0.51 - 0.17 = 0.62 - 0.28$  $0.51 < 0.62$ \[ C(0.51, 0.17) > C(0.62, 0.28) \]
Partial Ordering of BSC/SE: Summary

For two BSC/SEs \((a_1, b_1)\) and \((a_2, a_2)\), if

\[
\begin{align*}
    a_2 &< \frac{1 + a_1 - b_1}{2} \\
    b_2 &> b_1 \\
    a_2 - b_2 &< a_1 - b_1
\end{align*}
\]

then \(C(a_1, b_1) \geq C(a_2, b_2)\)
SR in BSC/SE

\[ Input \ x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

\[ n \sim N(0, \sigma^2) \]

\[ y = x + n \]

\[ t(\theta) = \begin{cases} 
-1 & x + n < -\theta \\
0 & -\theta \leq x + n \leq \theta \\
1 & x + n > \theta 
\end{cases} \]

\[ F_n(\theta) = P_r(n \leq \theta) \]
Review of Gaussian, Laplace, Cauchy noise

\begin{align*}
  f(n) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{n^2}{2\sigma^2}} \\
  F_n(\theta) &= P_r(n \leq \theta) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\theta} e^{-\frac{t^2}{2\sigma^2}} dt
\end{align*}

\begin{align*}
  f(n) &= \frac{1}{2\beta} e^{-\frac{|n|}{\beta}} \\
  F_n(\theta) &= P_r(n \leq \theta) = \begin{cases} 
    \frac{1}{2} e^{\frac{\theta}{\beta}} & \theta < 0 \\
    1 - \frac{1}{2} e^{-\frac{\theta}{\beta}} & \theta \geq 0
  \end{cases}
\end{align*}

\begin{align*}
  f(n) &= \frac{1}{\pi n^2 + \gamma^2} \\
  F_n(\theta) &= \frac{1}{\pi} \arctan \left( \frac{\theta}{\gamma} \right) + \frac{1}{2}
\end{align*}
SR in BSC/SE

\[
\begin{align*}
    p_{11} &= P(1|1) = 1 - F_N(-1 + \theta) \\
    p_{21} &= P(0|1) = F_N(-1 + \theta) - F_N(-1 - \theta) \\
    p_{31} &= P(-1|1) = F_N(-1 - \theta) \\
    p_{12} &= P(1|-1) = 1 - F_N(1 + \theta) \\
    p_{22} &= P(0|-1) = F_N(1 + \theta) - F_N(1 - \theta) \\
    p_{32} &= P(-1|-1) = F_N(1 - \theta)
\end{align*}
\]

\[
p_{31} = p_{12} \quad p_{21} = p_{22} \quad p_{11} = p_{32} \quad p_{11} > p_{12}
\]

\[
C(X,Y) = F_N(1 - \theta) \log_2 \left( \frac{F_N(1-\theta)}{0.5(F_N(1-\theta) - F_N(-1-\theta)) + F_N(-1-\theta)} \right) + \\
F_N(-1 - \theta) \log_2 \left( \frac{F_N(-1-\theta)}{0.5(F_N(-1-\theta) - F_N(1-\theta)) + F_N(1-\theta)} \right)
\]
SR in BSC/SE: Results
SR in BSC/SE-Gaussian Noise
SR in BSC/SE- Gaussian Noise
Forbidden Interval

Definition 5.2.1 [21] A threshold system exhibits stochastic resonance (as a function of \( \sigma \)) if and only if there exists \( 0 < \sigma_1 \) and \( 0 < \sigma_2 \) such that \( \sigma_1 < \sigma_2 \) and \( C_{\theta}(\sigma_1) < C_{\theta}(\sigma_2) \)

Theorem 5.2.1: The BSC/SE exhibits stochastic resonance if and only if the threshold \( \theta \notin [0,1] \).
SR and Channel Capacity Bounds

Pinsker Lower (L) and Helgert Upper (U) bounds

\[ M = \begin{pmatrix} p_{1,1} & \cdots & p_{n-1,1} & p_{n,1} \\ p_{1,2} & \cdots & p_{n-1,2} & p_{n,2} \end{pmatrix} \]

\[ p_{n,1} = 1 - p_{1,1} - \cdots - p_{n-1,1} \]
\[ p_{n,2} = 1 - p_{1,2} - \cdots - p_{n-1,2} \]

\[ L \leq C \leq U \]

For the (2,n) DMC

\[ L = \frac{1}{8 \ln(2)} \left[ \left( \sum_{i=1}^{n-1} |p_{i,1} - p_{i,2}| \right) + |p_{n,1} - p_{n,2}| \right]^2 \]

\[ U = \max \left( \sum_{i=1}^{n-1} p_{i,1} ; \sum_{i=1}^{n-1} p_{i,2} \right) - \left( \sum_{i=1}^{n-1} \min (p_{i,1} ; p_{i,2}) \right) \]

For \( n=3 \)

\[ L = \frac{1}{2 \ln(2)} \left( F_N(1 - \theta) - F_N(-1 - \theta) \right)^2 \]

\[ U = F_N(1 - \theta) - F_N(-1 - \theta) \]
SR and Helgert Bound
SR and Channel Capacity Bounds

[Graphs showing the relationship between power and channel capacity for different values of theta (0.5, 1, 1.5, 2).]
Optimal Values for Threshold and Noise Level

critical points of the capacity bounds

\[
\frac{d}{d\sigma} L(\theta, \sigma) = \frac{1}{\ln(2)} \left( F_N(1 - \theta) - F_N(-1 - \theta) \right) \left( F_N(1 - \theta) - F_N(-1 - \theta) \right) = \frac{1}{\ln(2)} \frac{d}{d\sigma} U(\theta, \sigma) \left( F_N(1 - \theta) - F_N(-1 - \theta) \right) \\
\frac{d}{d\sigma} L(\theta, \sigma) = 0 \quad \Longleftrightarrow \quad \frac{d}{d\sigma} U(\theta, \sigma) = 0
\]

\[
\frac{d}{d\sigma} U(\theta, \sigma) = \frac{d}{d\sigma} \left( F_N(1 - \theta) - F_N(-1 - \theta) \right) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-1-\theta}^{1-\theta} e^{-\frac{t^2}{2\sigma^2}} dt = \frac{-1}{\sigma^2 \sqrt{2\pi}} \left[ (1 - \theta) e^{\frac{-(1-\theta)^2}{2\sigma^2}} - (-1 - \theta) e^{\frac{-(1-\theta)^2}{2\sigma^2}} \right]
\]

\[
\frac{d}{d\sigma} U(\theta, \sigma) = 0 \quad \Longleftrightarrow \quad \sigma = \sqrt{\frac{2\theta}{\ln \left( \frac{1+\theta}{-1+\theta} \right)}}
\]
Optimal Values for Threshold and Noise Level

![Graph showing optimal values for threshold and noise level.](image-url)
Conclusion

- We analyzed the general (2,3) DMC and established the relations between the BAC/SE, BSC/SE, BSC and the BEC.

- We provided a partial ordering for the BSC/SE under specific conditional probability between channel input and channel outputs constraints using directional derivative and majorization theory.

- We studied the threshold based stochastic resonance behavior of binary-input ternary-output DMC in presence of Gaussian, Laplace and Cauchy noise.

- We demonstrated that in presence of the noise with even probability distribution function the general (2,3) DMC become the BSC/SE.

- We used the lower and upper capacity bounds to illustrated the stochastic resonance.

- We derived analytically the optimum noise power level depending on the threshold level that will maximize their capacity.
References

References (cont.)

- Paul Cotae and T.C. Yang “A cyclostationary blind Doppler estimation method for underwater acoustic communications using direct-sequence spread spectrum signals,” 8th International Conf. on Communications, COMM 2010, Bucharest, July 2010.
- Roland Kamdem, Paul Cotae and Ira S. Moskowitz, ”Threshold Based Stochastic Resonance for the Binary-Input Ternary-Output Discrete Memoryless Channel,” accepted at 7th IASTED International Conf. on Communication, Internet and Information Technology, CIIT 2012, Baltimore, May 2012.